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A NONLINEAR STRESS–STRAIN RELATION

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Abstract—A new nonlinear stress–strain relation for an isotropic, homogeneous material is presented. It provides a realistic nonlinear uniaxial stress–strain curve and accounts for the apparent variation of Poisson's ratio found experimentally when classical linear theory is assumed. It also accounts for the Poynting effect, for the nonlinear effect of hydrostatic stress superimposed upon simple tension and upon simple shear. The conditions set forth by Truesdell for the acceptability of a constitutive equation are met, and the nonlinear stress–strain relation reduces to the classical linear stress–strain relation for sufficiently small stresses.

1. INTRODUCTION

APPLIED loads and their consequent displacements are the fundamental laboratory quantities which characterize the elastic or the anelastic behavior of a material. However, these properties are conventionally described by relations between the derived quantities of stress and strain in order to state the constitutive properties of a material in a manner which is independent of any particular configuration and any particular loading.

In most theoretical analyses the stress is taken to be the actual force per unit of actual area which acts across a surface, while in many experimental works it is approximated by the force per unit of undeformed area across a reference surface in the initial configuration of the body. Experimentalists often refer to this approximate stress simply as stress, and refer to the theoretical stress as the true, or actual, stress. In what follows the term stress is to denote the true stress.

Although strain has been defined in a number of ways, we shall be concerned with only the more widely accepted definitions, referred to as classical strain, logarithmic (or "true") strain, Green–St. Venant, and Alamansi–Hamel strain.

Suitable nonlinear strain–displacement equations for the solution of the problem of large displacements of an elastic body have been formulated in Cartesian coordinates by Love [1] and by Novozhilov [2] in terms of the Green–St. Venant strain measure. These equations have been used by Green and Adkins [3] to determine the stresses which result from certain prescribed deformations of an elastic body. Corresponding equations in curvilinear coordinates appropriate to the Alamansi–Hamel strain measure have been given by Truesdell [4], but no explicit solutions involving these equations for an elastic material have been found in the literature.

Even though our main purpose is to present a nonlinear stress–strain relation which appears to be in good qualitative agreement with experimental observations, we shall digress in Sections 2 and 3 to demonstrate that the classical linear theory may not be extended to the case of finite deformation due to large stresses by retaining the linear stress–strain relations and simply replacing the classical linear strain–displacement relations with nonlinear strain–displacement relations. A brief discussion of logarithmic strain is presented in Section 4.

Motivated by the different deficiencies found in the linear theories when compared with actual observation, regardless of which of these four strain measures is used, we propose a nonlinear stress–strain relation in terms of general curvilinear tensors in Section 5. Not only may this three dimensional stress–strain relation include the uniaxial predictions of the logarithmic strain, but it is shown in Sections 6 through 10 that it also accounts for the apparent variation of Poisson's ratio in terms of the linear theory, for the Poynting effect, and for the increased shear strain due to a given shear stress in the presence of a superimposed hydrostatic stress. It also accounts for the recent results reported by Chalupnik and Ripperger [5]. In these sections it is to be understood that the strain measures are to be either the Green–St. Venant or the Alamansi–Hamel strain measures, depending upon the coordinate system chosen.

The initial assumptions herein differ from those given earlier by Truesdell [4], and by Huston [6] in that the form of the stress–strain relations is not based upon the assumption that the stress tensor is a function of the strain tensor, the strain tensor inner product, and the principle strain invariants, as given by Reiner [7] and others. Instead, it is based upon the assumption, also given by Reiner [7], that the strain tensor is a function of the stress tensor, its inner products, and its principle invariants, which seems to correspond more closely to the empirical relations proposed by Ludwik [8], by MacGregor [9], and by Timoshenko [10]. The method also differs from a number of recent investigations in that the existence of a strain–energy function is not assumed, in that arbitrarily large displacements are permissible, and in that explicit results are given in terms of specific constants. However, the significance of the results lies not in these differences, but rather in the display of stress–strain relations which do agree with experimental data and which also conform to the mathematical requirements set forth by Truesdell [4]. Consequently, the empirical, uniaxial relations advocated by Ludwik [8], MacGregor [9], and others, need no longer be considered isolated from the theory expressed by Truesdell [4, 11].

2. CLASSICAL LINEAR THEORY

In rectangular Cartesian coordinates the classical statement of the problem of the extension of a rod due to axial tension is that

$$2\varepsilon_{ij} = u_{i,j}+u_{j,i} \tag{1}$$

$$\varepsilon^{i}_{.j} = [(1+\nu)/E]\sigma^{i}_{.j}-(\nu/E)J_1\delta^{i}_{j} \tag{2}$$

$$J_1 \equiv \sigma^{i}_{.i} \tag{3}$$

$$\sigma_{ji,j}+F_i = 0 \quad \text{within the solid} \tag{4}$$

$$\sigma_i = \sigma_{ji}n_j \quad \text{on the bounding surface} \tag{5}$$

where σ_{ij} and ε_{ij} are the stress and strain tensors respectively, n_j is the outward normal to the bounding surface, E is Young's modulus, and ν is Poisson's ratio. The summation convention is assumed for repeated indices, and a comma implies partial differentiation with respect to the coordinate whose index follows. The x^3 axis is taken to lie along the axis of the rod, so that $u_3(x^3)$ is the displacement of the rod in the positive x^3 direction. Likewise, $u_1(x^1)$ and $u_2(x^2)$ are the displacements in the positive x^1 and x^2 directions respectively.

Body forces F_i will be assumed to vanish. Surface tractions are to act only over the ends of the rod, so that $\sigma_3 = \sigma_{33} = \sigma$ and $\sigma_1 = \sigma_2 = 0$ over the ends. Hence the only non-vanishing components of the strain are

$$\varepsilon_{11} = \varepsilon_{22} = -(\nu/E)\sigma \tag{6}$$

$$\varepsilon_{33} = \sigma/E \tag{7}$$

which yield the well-known results that

$$u_3 = (\sigma/E)x^3 \tag{8}$$

$$\left.\begin{matrix} u_1 \\ u_2 \end{matrix}\right\} = -(\nu/E)\sigma \left\{\begin{matrix} x^1 \\ x^2 \end{matrix}\right. \tag{9}$$

where $u_1(0) = u_2(0) = 0$. It is of interest to note that if we consider a circular cylindrical rod whose unstressed area is A_0 that its stressed area now is given by

$$A = A_0[1-(\nu/E)\sigma]^2, \tag{10}$$

from which the volume of the deformed rod may be computed. Thus, when $\sigma = E/\nu$ the area of the rod goes to zero, even though the length of the rod becomes $L = L_0(1+1/\nu)$, where L_0 is the length of the unstrained rod. These equations imply that a rod of incompressible material, $\rho = \rho_0$ and $\nu = 0{\cdot}50$, will vanish when stretched to three times its original length. Thus it is clear that in this case the classical linear theory of elasticity violates the macroscopic laws of conservation of matter, which is that $\rho_1 V_1 = \rho_2 V_2$, where ρ_1, ρ_2, V_1, and V_2 are the mass density and volume configurations 1 and 2 of a reference mass, or body.

In infinitesimal elasticity $(\sigma/E)x^3 = u_3$ is taken to be so small that the second and higher order terms may be neglected, so that

$$AL = [1-(\nu/E)\sigma]^2(1+\sigma/E)A_0L_0 \simeq [1+(1-2\nu)\sigma/E]A_0L_0,$$

which yields $AL = A_0L_0$ when $\nu = 1/2$. Second order violation of the conservation of matter is usually ignored in classical linear elasticity because the theory is valid only for vanishingly small displacements. Nevertheless, the violation does exist, and becomes important if we wish to examine non-infinitesimal displacements.

3. NONLINEAR STRAIN MEASURES

In this section we shall attempt to resolve the unrealistic results of the classical theory by retaining the classical linear stress–strain relations and introducing nonlinear strain–displacement relations. We shall begin with the Green–St. Venant strain–displacement relations [4] in rectangular Cartesian coordinates; namely,

$$2\varepsilon_{ij} = u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}. \tag{11}$$

Large deformations of a linearly elastic rod may then be described by equations (11) and equations (2) through (5). It is easy to show that these equations are satisfied by

$$\left.\begin{matrix} u_1 \\ u_2 \end{matrix}\right\} = (\sqrt{[1-2(\nu/E)\sigma]}-1)\left\{\begin{matrix} x^1 \\ x^2 \end{matrix}\right. \tag{12}$$

$$u_3 = (\sqrt{[1+2(\sigma/E)]}-1)x^3. \tag{13}$$

Strained and unstrained areas are now related according to

$$A = A_0[1-2(\nu/E)\sigma], \tag{14}$$

so that the area goes to zero as the length goes to $L_0\sqrt{(1+1/\nu)}$. Hence for materials with $\nu = 0{\cdot}50$ the specimen must vanish when it is stretched to $\sqrt{3}$ times its original length. Incompressible matter is, therefore, either destroyed or compressed to infinite density under the action of axial tension.

In the previous strain measure the strains and displacements were expressed in terms of a coordinate system which was fixed relative to the initial, or unstrained, configuration. The Alamansi–Hamel strain measure [4] differs from the previous strain measure in that it is defined in terms of a coordinate system which usually is associated with the final, or strained, configuration. Consequently, the strain–displacement relations in Cartesian coordinates are now given by

$$2\varepsilon_{ij} = u_{i,j}+u_{j,i}-u_{k,i}u_{k,j} \tag{15}$$

and the displacements are given by

$$\left.\begin{matrix} u_1 \\ u_2 \end{matrix}\right\} = \{1-\sqrt{[1+2(\nu/E)\sigma]}\}\left\{\begin{matrix} x_1 \\ x_2 \end{matrix}\right. \tag{16}$$

$$u_3 = \{1-\sqrt{[1-2(\sigma/E)]}\}x^3 \tag{17}$$

which satisfy equations (15), (2) through (5), and the boundary conditions. Since the x^i coordinate system has been selected to describe the strained configuration, the original, or unstrained length of the rod in question may be found in terms of the strained length from

$$L_0 = L-u^3(L) = \sqrt{[1-2(\sigma/E)]} \tag{18}$$

so that

$$L = L_0/\sqrt{[1-2(\sigma/E)]} \tag{19}$$

$$A = A_0/[1+2(\nu/E)\sigma]. \tag{20}$$

Together equations (19) and (20) imply that the volume of an incompressible rod must increase when the bar is placed in tension. That is, equation (19) requires the rod to become infinitely long as the tensile stress approaches $E/2$, while equation (20) demands that the cross-sectional area of an incompressible rod approach $2A_0/3$.

4. LOGARITHMIC STRAIN

Logarithmic strain appears to have been suggested for one dimensional problems as the result of a feeling that the classical strain measures over-emphasized the role of the initial length in the case of non-infinitesimal deformation [8–10]. In this empirical modification of the theory the strain–displacement relations (1) are replaced by

$$\varepsilon' = \log(1+\varepsilon) \tag{21}$$

where $\varepsilon = (L - L_0)/L_0$. By turning to the stress–strain relation

$$\varepsilon' = \sigma/E \tag{22}$$

usually associated with the logarithmic strain [8–10], and then substituting for ε' according to equation (21), it is easy to see that equation (22) is equivalent to

$$\varepsilon = e^{\sigma/E} - 1. \tag{23}$$

Introduction of the logarithmic strain measure thus means that we have actually replaced the algebraically linear stress–strain relation (2) by the nonlinear stress–strain relation (23), in terms of the classical measure of strain which appears in equation (2).

Replacement of equation (2) by the single equation (23) demands that lateral distortion be neglected. Advocates of equation (22), or equation (23), have, therefore, adjoined the condition that

$$\varepsilon' = -\log(1-q) \tag{24}$$

where $q = (A_0 - A)/A_0$. Since equation (23) and (24) are equivalent to

$$L = L_0\, e^{\sigma/E} \tag{25}$$

$$A = A_0\, e^{-\sigma/E} \tag{26}$$

it is obvious that incompressible matter is conserved in this theory.

Objections to the use of logarithmic strain are that (i) the constant volume relationship implied by equations (25) and (26) may not be realistic for large strains, and that (ii) it has been achieved by neglecting Poisson's ratio. Moreover, it has been held that (iii) no satisfactory extension of the logarithmic strain measure to three-dimensional deformation has been found [4].

Nevertheless, numerous arguments in favor of logarithmic strain, as opposed to classical strain, have appeared in the literature, all with the central theme that better agreement is achieved between theory and experiment when logarithmic strain is used. This may be expected because of the similar nonlinearity of equation (23) to that of the usual uniaxial stress–strain curve.

5. AN IMPROVED NONLINEAR STRESS–STRAIN RELATION

Inasmuch as neither the Green–St. Venant nor the Alamansi–Hamel strain measures improved the agreement between theoretical and experimental results when linear stress–strain relations were employed, it seems reasonable to replace the linear relations (2) with some nonlinear form. Huston [6] has shown that if terms of the third order and higher are dropped, then the stress tensor in an isotropic, homogeneous, elastic continuum may be given in terms of the strain tensor as

$$\sigma^i_{.j} = \frac{E}{(1+\nu)(1-2\nu)}[(I_1 - 2I_2)\nu\delta^i_j + (1-2\nu)\varepsilon^i_{.j} + (5\nu - 1)\varepsilon^i_{.k}\varepsilon^k_{.j}], \tag{27}$$

where $I_1 = \varepsilon^i_{.i}$ and $I_2 = \varepsilon^i_{.k}\varepsilon^k_{.i}$. According to these equations, however, a circular cylinder of incompressible material cannot deform such that $\varepsilon^1_{.1} \neq 0$ and $\varepsilon^2_{.2} = \varepsilon^3_{.3} \neq 0$ when subjected to a uniaxial tension or compression in which the only nonvanishing stress component is finite. That is, a material which obeys equation (27) is perfectly rigid under axial loading when $\nu = 0{\cdot}50$. This conflicts with experimental data when the reduction of cross-sectional area is taken into account [12].

Based upon the generality of Huston's interesting result we conclude that stress–strain relations which display a more realistic behavior may be obtained if we assume from the outset that the strain may be expressed as a function of the stress tensor and its inner products, as

$$\varepsilon^i_{.j} = A_1\delta^i_j + A_2\sigma^i_{.j} + A_3\sigma^i_{.k}\sigma^k_{.j} \tag{28}$$

in which $A_k = A_k(J_1, J_2, J_3, \bar{J}_2, \bar{J}_3)$ for $k = 1, 2, 3$ where the stress invariants are defined by

$$J_1 = \sigma^i_{.i} \qquad \bar{J}_1 = \sigma^i_{.i}$$

$$J_2 = \frac{1}{2!}\delta^{ij}_{mn}\sigma^m_{.i}\sigma^n_{.j} \qquad \bar{J}_2 = \sigma^i_{.k}\sigma^k_{.i}$$

$$J_3 = \frac{1}{3!}\delta^{ijk}_{mnp}\sigma^m_{.i}\sigma^n_{.j}\sigma^p_{.k} \qquad \bar{J}_3 = \sigma^i_{.j}\sigma^j_{.k}\sigma^k_{.i}$$

These two sets of stress invariants are not independent, but are related according to $\bar{J}_3 = J_1(\bar{J}_2 - J_2) + 3J_3$. This representation is distinctly different from the representation assumed in the derivation of equation (27); namely, that

$$\sigma^i_{.j} = B_1\delta^i_j + B_2\varepsilon^i_{.j} + B_3\varepsilon^i_{.k}\varepsilon^k_{.j} \tag{29}$$

because equation (29) cannot generally be solved explicitly for $\varepsilon^i_{.j}$. This difference is of engineering importance because equation (28) can provide good agreement between theory and experiment, as will be shown.

In the case of uniaxial tension it is evident from the Hamilton–Cayley theorem that the coefficients A_1, A_2, and A_3 may be chosen such that equations (28) agree with the stress–strain curve obtained from the logarithmic strain relations (23) to whatever order is desired [13]. Relations (28), therefore, serve to bridge the gap which has existed between the logarithmic strain relations and the mathematical requirements given by Truesdell [11]. Because of the wide variety of choices that are possible for coefficients A_k, equations (28) may, in fact, provide a more realistic description of actual materials than may be obtained from previously suggested relations.

The remaining sections will be devoted to the nonlinear stress–strain relation

$$\varepsilon^i_{.j} = [\alpha\bar{J}_3 + \gamma\bar{J}_2 - (\nu/E)J_1]\delta^i_j + [(1+\nu)/E]\sigma^i_{.j} - \eta\sigma^i_{.k}\sigma^k_{.j} \tag{30}$$

which is obtained from equation (28) when the A_k are chosen to agree with the first three terms of equation (23). It will be shown that this equation, which holds for isotropic, homogeneous, elastic materials is sufficient to account for a number of actual phenomena not included in those theories based upon equations of the form of equation (29).

Evans and Pister [14] have obtained a stress–strain relation similar to equation (30) from the complementary energy density function, but subject to the condition of classical infinitesimal strain. Since stress–strain relation (30) is a special case of relation (28), valid for large strains [4, 7, 11], it may be viewed as an extension of the work of Evans and Pister to include large strains. The coefficients in equation (30) have been selected such that the isotropy group is the full orthogonal group [10, 15]. These constants have the dimensions $\alpha \sim (LT^2/M)^3$; $\gamma, \eta \sim (LT^2/M)^2$; $E \sim M/(LT^2)$; and $\nu \sim 1$. Representative values of these coefficients will be determined in the next section.

6. UNIAXIAL TENSION AND POISSON'S RATIO

Let $\sigma^1_{.1} = \sigma$ be the only nonvanishing stress component. Then the nonzero stress invariants are $J_1 = \sigma$, $\bar{J}_2 = \sigma^2$, and $\bar{J}_3 = \sigma^3$, so that

$$\varepsilon^i_{.j} = [\alpha\sigma^3+\gamma\sigma^2-(\nu/E)\sigma]\delta^i_j+[(1+\nu)/E]\sigma^i_{.j}-\eta\sigma^i_{.k}\sigma^k_{.j}. \tag{31}$$

Consequently

$$\varepsilon^1_{.1} = \sigma/E+(\gamma-\eta)\sigma^2+\alpha\sigma^3 \tag{32}$$

$$\varepsilon^2_{.2} = \varepsilon^3_{.3} = [\alpha\sigma^3+\gamma\sigma^2-(\nu/E)\sigma] \tag{33}$$

It is at this point that the coefficients $(\gamma-\eta)$ and α may be specified such that equation (32) will agree with the first three terms of the series expansion of the stress–logarithmic strain relation (23). In the remainder of the discussion we shall assume the values $\nu = 0{\cdot}25$, $E = 3{\cdot}0\times 10^7$ psi, $\alpha = 6{\cdot}173\times 10^{-24}$ in^6/lb^3, $(\gamma-\eta) = 5{\cdot}556\times 10^{-16}$ in^4/lb^2. Particular values of γ and η may be chosen from a shear test or from combined shear and hydrostatic loading.

From equation (32) and (33) we find that the ratio of the lateral contraction to the negative of the longitudinal extension becomes

$$\nu' = -\frac{\alpha\sigma^3+\gamma\sigma^2-(\nu/E)\sigma}{\sigma/E+(\gamma-\eta)\sigma^2+\alpha\sigma^3} \tag{34}$$

which reduces to the classical value of Poisson's ratio ν when the longitudinal stress is sufficiently small. This ratio ν' is in good agreement with the experimentally determined ratio reported by Goodman [16] for aluminum and titanium alloys.

7. HYDROSTATIC LOADING

A hydrostatic load corresponds to the stress field $\sigma^1_{.1} = \sigma^2_{.2} = \sigma^3_{.3} = -\sigma$ where all other stress components vanish. In this event $J_1 = -3\sigma$, $\bar{J}_2 = 3\sigma^2$, and $\bar{J}_3 = -3\sigma^3$, so that

$$\varepsilon^1_{.1} = \varepsilon^2_{.2} = \varepsilon^3_{.3} = -3\alpha\sigma^3+(3\gamma-\eta)\sigma^2-[(1-2\nu)/E]\sigma. \tag{35}$$

A material which obeys equation (30) is, as a result of equation (35), incompressible if $\alpha = 0$, $3\gamma = \eta$, and $\nu = 0{\cdot}50$. In contrast to the classical linear theory, we find that nonlinear materials represented by equation (30) cannot be characterized as incompressible solely on the value of Poisson's ratio. Furthermore, according to the proposed theory incompressible behavior under hydrostatic loading does not imply incompressible behavior under other loading conditions.

8. COMBINED SHEAR AND HYDROSTATIC STRESS

To examine the effect of hydrostatic loading upon the strain response of a material described by equation (30) we assume that $\sigma^1_{.1} = \sigma^2_{.2} = \sigma^3_{.3} = -\sigma$ and that $\sigma^1_{.2} = \sigma^2_{.1} = \tau$ with stresses $\sigma^1_{.3} = \sigma^3_{.1} = \sigma^2_{.3} = \sigma^3_{.2} = 0$ in terms of a rectangular Cartesian frame. Then

$$J_1 = \bar{J}_1 = -3\sigma, \qquad \bar{J}_2 = 3\sigma^2+2\tau^2, \qquad \bar{J}_3 = -3\sigma^2-6\sigma\tau^2 \tag{36}$$

$$\varepsilon^i_{.j} = [3\alpha\sigma^3-6\alpha\sigma\tau^2+3\gamma\sigma^2+2\gamma\tau^2+3(\nu/E)\sigma]\delta^i_j+[(1+\nu)/E]\sigma^i_{.j}-\eta\sigma^i_{.k}\sigma^k_{.j}. \tag{37}$$

Upon setting $i = 1$, and $j = 2$, we find that

$$\varepsilon^{1}_{.2} = [(1+\nu)/E + 2\eta\sigma]\tau \tag{38}$$

in which the strain due to a given value of the shear stress τ increases with increasing hydrostatic stress because of the second term within square brackets in equation (38). Because η is of the order of 10^{-16} in^4/lb^2 while $(1+\nu)/E$ is of the order of 10^{-7} in^2/lb, the second term will become important only in the presence of large hydrostatic forces. An effect of this nature has been reported by Bridgman [17].

9. COMBINED TENSION AND HYDROSTATIC STRESS

The effect of a hydrostatic pressure $-p$ superimposed upon simple tension may be realized with the aid of relations (30) and

$$J_1 = -3p+\sigma, \qquad \bar{J}_2 = 3p^2-2p\sigma+\sigma^2, \qquad \bar{J}_3 = 3p^3+3p^2\sigma-3p\sigma^2+\sigma^3$$

where $\sigma^{1}_{.1} = -p+\sigma, \sigma^{2}_{.2} = \sigma^{3}_{.3} = -p$. The resulting expression for the longitudinal strain is

$$\begin{aligned}\varepsilon^{1}_{.1} = &-\{3\alpha p^3-(3\gamma-\eta)p^2+[(1-2\nu)/E]p+3\alpha(\sigma-p)\sigma p+2(\gamma-\eta)p\sigma\} \\ &+\alpha\sigma^3+(\gamma-\eta)\sigma^2+\sigma/E.\end{aligned} \tag{39}$$

Since the square bracket is positive for the values of α, γ, and η previously chosen from the uniaxial test, it follows that the longitudinal strain decreases as the hydrostatic pressure increases. This result agrees well with the static test results published recently by Chalupnik and Ripperger [5].

10. TORSION

In this section we shall first demonstrate that the conventionally assumed stress field $\sigma_{\theta z} = r\tau$, with all other stresses equal to zero, which is satisfactory in the linear theory, fails to produce physically plausible results when the higher order terms which appear in equation (30) are retained. We then show that no such difficulties arise if the stresses σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$, σ_{zz}, in terms of a circular cylindrical coordinate system, are assumed to be nonzero. It follows from equation (30) that these stresses give rise to the Poynting effect and to the interior stress field reported by Nadai [18].

We begin by assuming that the only nonvanishing tensor components of the stress field are $\sigma^{2}_{.3} = \tau =$ const., and $\sigma^{3}_{.2} = (x^1)^2\tau$ when expressed in circular cylindrical coordinates where $x^1 = r$, $x^2 = \theta$, and $x^3 = z$. The physical components of tensors $\sigma^{2}_{.3}$ and $\sigma^{3}_{.2}$ are identical to one another; namely, $\sigma_{\theta z} = r\tau$. Then $J_1 = 0$, $\bar{J}_2 = 2(x^1\tau)^2$, and $\bar{J}_3 = 0$. Substitution into equation (30) gives

$$\begin{aligned}\varepsilon^{1}_{.1} &= 2\gamma(x^1\tau)^2 \\ \varepsilon^{2}_{.2} = \varepsilon^{3}_{.3} &= (2\gamma-\eta)(x^1\tau)^2 \\ \varepsilon^{2}_{.3} &= [(1+\nu)/E]\tau\end{aligned}$$

and

$$c_{11} = 4\gamma(x^1\tau)^2+1 \qquad c_{33} = 2(2\gamma-\eta)(x^1\tau)^2+1$$
$$c_{22} = (x^1)^2[2(2\gamma-\eta)(x^1\tau)^2+1] \qquad c_{23} = 2[(1+\nu)/E](x^1)^2\tau$$

where $c_{ij} = 2\varepsilon_{ij}+g_{ij}$ in which g_{ij} is the metric tensor. The radius of the twisted rod then becomes

$$\bar{a} = \int_0^a (c_{11})^{\frac{1}{2}}\,dr = \frac{a}{2}(4\gamma a^2\tau^2+1)^{\frac{1}{2}}+\frac{1}{4\tau\gamma^{\frac{1}{2}}}\log[2\gamma^{\frac{1}{2}}\tau a+(4\gamma a^2\tau^2+1)^{\frac{1}{2}}]$$

and the circumference becomes

$$\bar{d} = \int_0^{2\pi} [c_{22}(a)]^{\frac{1}{2}}\,d\theta = 2\pi a[2(a\tau)^2(2\gamma-\eta)+1]^{\frac{1}{2}}$$

where a is the radius of the unstrained rod. Since it must still be true that $\bar{d} = 2\pi\bar{a}$, this relation can hold only for a particular value of γ/η. Since these results are to be independent of particular values of the parameters, we conclude that the classical fields cannot be generally realized.

On the other hand, the stress field

$$\begin{gathered}\sigma^1_{.1} = s(x^1) \qquad \sigma^2_{.2} = \sigma(x^1) \qquad \sigma^3_{.3} = p(x^1) \\ \sigma^2_{.3} = \tau(x^1) \qquad \sigma^3_{.2} = (x^1)^2\tau(x^1)\end{gathered} \tag{40}$$

whose physical components are

$$\sigma_{rr} = s(r), \qquad \sigma_{\theta\theta} = \sigma(r), \qquad \sigma_{zz} = p(r)$$
$$\sigma_{\theta z} = \sigma_{z\theta} = r\tau(r)$$

may be realized. These stresses satisfy the boundary conditions that $\sigma_{rr}(a) = 0$, and that the relations

$$T = 2\pi\int_0^a \sigma_{\theta z}r^2\,dr, \qquad \int_0^a \sigma_{zz}r\,dr = 0$$

hold over the ends of the bar, where T is the applied torque.

Evaluation of the pertinent stress invariants from display (40) yields

$$\bar{J}_1 = s+\sigma+p, \qquad \bar{J}_2 = s^2+\sigma^2+p^2+2(x^1\tau)^2, \qquad \bar{J}_3 = s^3+\sigma^3+p^3+3(x^1\tau)^2(\sigma+p).$$

Substitution of these expressions into equations (30) in turn yields

$$\begin{aligned}\varepsilon^1_{.2} &= \varepsilon^2_{.1} = \varepsilon^1_{.3} = \varepsilon^3_{.1} = 0 \\ \varepsilon^1_{.1} &= \alpha\bar{J}_3+\gamma\bar{J}_2+(1/E)[s-\nu(\sigma+p)]-\eta s^2 \\ \varepsilon^2_{.2} &= \alpha\bar{J}_3+\gamma\bar{J}_2+(1/E)[\sigma-\nu(s+p)]-\eta[\sigma^2+(x^1\tau)^2] \\ \varepsilon^2_{.3} &= [(1+\nu)/E]\tau-\eta(\sigma+p)\tau \\ \varepsilon^3_{.3} &= \alpha\bar{J}_3+\gamma\bar{J}_2+(1/E)[p-\nu(s+\sigma)]-\eta[p^2+(x^1\tau)^2]\end{aligned} \tag{41}$$

which are the stress–strain relations which govern the finite twisting of a rod.

The field equations governing s, p, σ, and τ may be formed from equations (41) and the equations of equilibrium, which reduce to

$$s_{,r} + s/r = \sigma/r \tag{42}$$

and the finite compatibility relations [12, 17]

$$\begin{aligned} R_{ijmn} &= \tfrac{1}{2}(c_{in,\,jm} + c_{jm,\,in} - c_{im,\,jn} - c_{jn,\,im}) \\ &\quad + c^{kp}([jm,k][in,p] - [jn,k][im,p]) = 0 \end{aligned} \tag{43}$$

where $c^{ij} = C^{ij}/c$ in which C^{ij} is the cofactor of the matrix element c_{ji}, $c = \det(c_{ij})$, and where $[ij, k]$ is the Christoffel symbol of the first kind.

For simplicity let $c_{11} = \zeta$, $c_{22} = \theta$, $c_{33} = \psi$, and $c_{23} = \varphi$, then the only nonvanishing components of the Riemann–Christoffel tensor R_{ijmn} may be written as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left\{\log\left[\frac{1}{(\mathrm{d}\theta/\mathrm{d}r)}(B\zeta)^{\frac{1}{2}}\right]\right\} = 0 \tag{44}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left\{\log\left[\frac{1}{(\mathrm{d}\varphi/\mathrm{d}r)}\left(\frac{\zeta B}{B_0}\right)^{\frac{1}{2}}\right]\right\} = 0 \tag{45}$$

$$-B\frac{1}{(\mathrm{d}\psi/\mathrm{d}r)}\frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} + B\frac{1}{\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}r} + \frac{1}{2}\left[\frac{\mathrm{d}B}{\mathrm{d}r} + (1-\psi)\frac{\mathrm{d}\theta}{\mathrm{d}r}\right] = 0 \tag{46}$$

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right)^2 - \frac{\mathrm{d}\theta}{\mathrm{d}r}\frac{\mathrm{d}\psi}{\mathrm{d}r} = 0 \tag{47}$$

where $B = \theta\psi - \varphi^2$. Equations (44) through (47) may be satisfied by

$$\varphi = c_1, \qquad \theta = c_2, \qquad \psi = c_3 \tag{48}$$

where $c_1^2 = c_2 c_3$. As a consequence, the first equation of set (48) may be written as

$$\tau = \frac{c_1 E}{2(1+\nu) - \eta E(\sigma + p)}\frac{1}{r^2} \tag{49}$$

Thus τ and θ in the second and third equations of (48) may be expressed in terms of s and p and their derivatives by means of equations (43) and (49). Numerical methods may be used to solve the resulting system of two simultaneous ordinary nonlinear differential equations. Rather than becoming engrossed in the numerical solution of these equations, however, we shall instead linearize equations (48) by neglecting terms having coefficients α, γ, or η in expressions (41) and (49). Although the neglect of these terms may not be entirely justified near the axis of the rod, the resulting expressions qualitatively verify that the Poynting effect is accounted for by stress–strain relations (41), and hence by equations (30), and that these equations also account for the large compressive stresses found by E. A. Davis along the axis of a twisted rod [18].

The linearized second equation of (48)

$$\theta = 2r^2(1/E)[\sigma - \nu(s+p)] + r^2 = c_2$$

may be satisfied by

$$s(r) = \frac{\nu}{r^{1-\nu}}\int p\frac{dr}{r^{\nu}} - \nu\frac{a^{1+\nu}}{r^{\nu}}\int^{a} p\frac{dr}{r^{\nu}} + \frac{E}{2(1-\nu)}\left[\left(\frac{a}{r}\right)^2 - 1\right] + \frac{c_4}{r^2}(a^{1+\nu} - r^{1+\nu}) \tag{50}$$

where $s(a) = 0$. Substitution from equation (50) into the third linearized equation of (48)

$$\frac{2}{E}[p - \nu r s_{,r} - 2\nu s] + 1 = c_3$$

leads to

$$p = \frac{E}{\nu(1+\nu)}\left[1 + \frac{1-c_3}{2\nu}\right] = 0 \tag{51}$$

due to the boundary conditions over the ends of the rod. Condition (51) can be satisfied only if

$$c_3 = 1 + 2\nu > 0. \tag{52}$$

Thus the length of the rod must increase. Consequently, the Poynting effect is included in equations (41) and hence in tensor equation (30). Dependence of the increase in length upon the applied torque has been suppressed by the neglect of α, γ, and η terms in equation (41).

Substitution for s from equation (50) into equation (43) and use of the boundary condition that $\sigma_{\theta\theta} =$ const. on the lateral surface permits the evaluation of a constant which also appears in the expression for σ_{rr}. The radial stress may then be written as

$$\sigma_{rr}(r) = -\frac{E}{2(1+\nu)}\left[\left(\frac{a}{r}\right)^2 - \frac{2}{1-\nu}\left(\frac{a}{r}\right)^{1-\nu} + \frac{1+\nu}{1-\nu}\right] \tag{53}$$

where the singularity at the axis implies a large compressive stress along the center line of a twisted rod. Replacement of a large value by a singularity and the independence of the compressive stress of the applied torque are again due to the linearization of equations (48). Nevertheless, these results are sufficient to show qualitative agreement between theory and experiment.

11. CONCLUSIONS

We have shown that tensor equation (30), which satisfies the seven conditions enunciated by Truesdell [12] for admissible constitutive equations, is adequate to realistically account for a number of phenomena which have not been explained by previous nonlinear stress–strain relations. Moreover, the agreement between theory and experiment for a number of loading conditions is achieved using three additional constants α, γ, and η, which may be determined from tension and shear tests only. With just these three additional constants determined from two tests we have accounted for five phenomena, so that it is reasonable to believe that the good agreement is not due to curve fitting, but rather due to the physical significance of the terms in equation (30). Because this agreement between theory and experiment rests upon the structure of relations (30), we have a nonlinear stress–strain relation that is simple enough to be of value in the engineering analysis of heavily loaded members.

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Абстракт—Приводится новая нелинейная зависимость между напряжением и деформацией для изотропного однородного материала. Это дает возможность представить в виде нелинейной соосной кривой наряжение-деформация и, выясняет видимое изменение соотношение Пуассона, найденного экспериментальным путем в случае учета классической линейной теории. Это также выясняет эффект Пойнтинга и нелинейный эффект гидростатического напряжения, накладываемого на простое растяжение и простый сдвиг. Условия, выведены впредь Трусделлом для приемлемости уравнения состояния, являются удовлетворенными и, нелинейная зависимость между напряжением и дефромацией превращается в классическую линейную зависимость между напряжением и деформацией для достаточно малых напряжений.